International Journal of Theoretical Physics, Vol. 40, No. 1, 2001

Dirac Operator on the Podleś Sphere[1]

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Received September 29, 1999

The spectral problem for the Dirac operator on the Podleś sphere is discussed and solved in one of its possible formulations. The standard constructions for the Dirac operator on classical symmetric spaces and the spectrum of the Dirac operator on the classical two-dimensional sphere are recalled. The problem of defining a spinor structure on a quantum space is discussed and the definitions of a classical spinor structure and Dirac operator according to Đurđevich are sketched. The Dirac operator for the Podleś quantum sphere treated as a quotient space of $S_\mu U(2)$ is constructed using the Woronowicz left-covariant calculus over this quantum group. The spectrum of the operator is obtained. Disagreement of its asymptotic behavior with Connes' axiom of noncommutative spectral geometry is stressed.

1. INTRODUCTION

Spectra of the Dirac operator on compact symmetric spaces play an important role in Kałuża–Klein theories (Appelquist *et al.*, 1987) and in condensed matter models (Makaruk, 1995, 1996). They are an alternative for the metric structure on Riemannian spin manifolds. They carry information about the topology of the spaces, as is shown in spectral noncommutative geometry introduced by Connes (1994).

The internal space is responsible for interactions observed in space-time and its size is usually assumed to be of the order of the Planck length, 10^{-33} m. The postulate that a space of this size has properties of a classical differential

[1] This work was undertaken under the auspices of the U.S. Department of Energy under contract W-7405-ENG-36. It was also supported partially by the Consejo Nacional de Ciencia y Tecnología (CONACyT) de México, Grant # 27670 E (1999–2000), and by the Universidad Nacional Autónoma de México, DGAPA, Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica, Proyecto IN-109599 (1999–2002).
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manifold is questionable. At such distances, it should be rather a kind of quantum space, to which the concept of points is no longer applicable. Such spaces are described in the framework of noncommutative geometry (Connes, 1994; Woronowicz, 1987; Đurđevich, 1996). Spectra of the Dirac operator on such spaces would give important information about the effective theory in physical space-time. Additionally, such spectra would be very interesting from the point of view of description of energy levels of the theories leading to the algebraic treatment of the spectra for classical symmetric spaces. For classical spheres, there exist a couple of interesting, yet quite different approaches to derivation of the spectrum of the Dirac operator: In one of them, the sphere is treated as a hypersurface in the ambient Euclidean space (Trautman, 1992, 1995). This approach enables derivation of the spectrum for a sphere of arbitrary dimension. Unfortunately, this approach seems not to be extendible to the case of quantum spheres. On the other hand, the approach treating spheres as symmetric spaces (Cahen and Gutt, 1988; Bär, 1992) looks much more plausible in the context of such generalization.

In Section 2, we recall, mainly following Cahen and Gutt (1988), how the standard spin structure and the Dirac operator look when introduced on classical symmetric spaces. We stress the algebraization of the operator and its spectral problem for such spaces due to so-called Wigner symmetry of such systems. We also review the form of the spectrum for the classical two-dimensional sphere. In Section 3, we discuss the generalization of the notion of spin structure and of the Dirac operator to the noncommutative context, following Đurđevich (1994), within the framework of classical spin structures, which is one such generalization. We also mention other possibilities, which we do not discuss in this paper. In this section, we also discuss possible generalizations of the notion of a symmetric space to the noncommutative context, pointing out difficulties with such notion, and finally limiting the further discussion to quotient quantum spaces. In Section 4, we introduce the Dirac operator and calculate its spectrum for the case of a quantum sphere (Podleś, 1987), treated as a quantum quotient space of the $S_\mu U(2)$ quantum group (Woronowicz, 1987), equipped with a classical spin structure, and with the Dirac operator coming from the left-covariant differential calculus on $S_\mu U(2)$ (Woronowicz, 1987). We stress disagreement of its asymptotic behavior with Connes' axiom concerning asymptotic of spectra of the Dirac operator on noncommutative spaces.

2. DIRAC OPERATOR AND ITS SPECTRUM ON CLASSICAL SYMMETRIC SPACES

The definition of a spin structure on a Riemannian oriented manifold (M, g) is given in Owczarek (1999). Let us discuss the constructions of spin structures on Riemannian, compact, simply connected, symmetric spaces represented as the quotient spaces $M = G/K$, where G is a compact and simply connected Lie group and $K = (G^{\sigma})_o$ is the connected component of the identity of the group of fixed points of σ, with σ an involutive automorphism of G. Let $\mathcal{G}$ be the Lie algebra of G, $\mathcal{K}$ the Lie algebra of K, and $\mathcal{P} = \{x \in \mathcal{G}: \sigma_{*e}\, x = -x\}$, where e is the neutral element of G and $\sigma_{*e}: \mathcal{G} \rightarrow \mathcal{G}$ is the automorphism of $\mathcal{G}$ derived from σ. The vector space $\mathcal{P}$ is isomorphic with the tangent space to M at $o = p(e)$, where $p: G \rightarrow G/K$ is the canonical projection. Moreover, T_oM with the metric $g|_{T_oM}$ is, up to a homothety, isometric with $\mathcal{P}$ equipped with the Killing form of $\mathcal{G}$ limited to $\mathcal{P}$. Cahen and Gutt (1988) showed that the orthonormal frame bundle over $M = G/K$ can be identified with $F(M) = G \times_{\alpha} SO(m)$, where $\alpha: K \rightarrow SO(m)$ is the homomorphism derived from the natural left action of G on M and the subscript α means that we consider equivalence classes defined by $[g, A] = [gk, \alpha(k^{-1})A]$, where $k \in K$, $g \in G$, $A \in SO(m)$. The derivative of the latter action at a point of M acts isometrically in the tangent space to M at this point. Construction of spin structures on these spaces depends on the existence of lifts of the homomorphisms α discussed above to homomorphisms α': $K \rightarrow \mathrm{Spin}(m)$, since one can easily prove (Cahen and Gutt, 1988; Bär, 1992) that the mapping from the set of lifts α' of α to the set of spin structures on M is a 1–1 correspondence. As a result, the form of the spin structure on M is $\tilde{F}(M) = G \times_{\alpha'} \mathrm{Spin}(m)$, where the subscript α' plays an analogous role to the subscript α above. With this principal $\mathrm{Spin}(m)$ bundle is associated, through the standard representation $\rho_m: \mathrm{Spin}(m) \rightarrow \mathrm{End}(\Sigma_m)$ of the spin group $\mathrm{Spin}(m)$ in the spinor space Σ_m, the vector bundle of spinors, which can be written as $\mathcal{S} = G \times_{\rho_m\alpha'} \Sigma_m$. On the space of smooth sections of this bundle (the space of smooth spinor fields), which can be understood as the tensor product of the space of $\mathbb{C}$-valued functions on the group G by Σ_m, with appropriate identifications, one can define a natural scalar product using the Haar measure on G and the natural scalar product in Σ_m. Let the completion of the space of smooth sections of $\mathcal{S}$, $C^{\infty}(G, \Sigma_m)_K$, where the subscript K means equivariance identification of elements of $C^{\infty}(G, \Sigma_m)$, in the norm defined by this scalar product be denoted $L^2(G, \Sigma_m)_K$. The Dirac operator can be extended from $C^{\infty}(G, \Sigma_m)_K$ to a self-adjoint operator on $L^2(G, \Sigma_m)_K$. As a consequence of the Peter–Weyl theorem, we have Frobenius reciprocity,

$$L^2(G, \Sigma_m)_K = \bigoplus_{\gamma \in \hat{G}} V_{\gamma} \otimes \mathrm{Hom}(V_{\gamma}, \Sigma_m)_K \tag{1}$$

where $\hat{G}$ is the unitary dual of G and $(V_{\gamma}, \rho_{\gamma})$ is a representative of the equivalence class γ. This theorem is very important for algebraization of the Dirac operator on the symmetric spaces since D commutes with the left action of G, induced from its natural action on a symmetric space, viewed as a homogeneous space of G. This property is strictly related to the fact that the Dirac operator is constructed using the Levi-Civita connection, which is expressed in terms of horizontal left-invariant vector fields on G, identifiable with elements of $\mathcal{G}$. Therefore, the Dirac operators on symmetric spaces satisfy the requirements for systems with a Wigner symmetry (in this case with the symmetry group G), as they are described, e.g., in Hurt and Hermann (1980). For such systems, the spectral problem is reduced to the purely algebraic problem connected with finite-dimensional unitary irreducible representations of the symmetry group. Our constructions in the noncommutative case will be also directed toward obtaining a system with Wigner symmetry, this time of a quantum group instead of a classical one. Then the knowledge of representations (unitary irreducible) of this quantum group is enough to derive the desired spectrum (at least in principle, since the algebraic calculations still might be quite complicated in the general case). As a result of the commutation of the operator D with the left $SU(2)$ action, D stabilizes the subspaces defined by the equality (1). The identification of elements of $V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_m)_K$ with elements of $C^\infty(G, \Sigma_m)_K$ by

$$(v \otimes A)(g) = A(\rho_\gamma(g^{-1})v), \qquad v \in V_\gamma, \quad A \in \mathrm{Hom}(V_\gamma, \Sigma_m)_K$$

shows that elements of $V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_m)_K$ are in the domain of D. If $\widetilde{D_\gamma}$ is the restriction of D to $V_\gamma \otimes \mathrm{Hom}(V_\gamma, \Sigma_m)_K$, then $\widetilde{D_\gamma} = id \otimes D_\gamma$, with

$$D_\gamma(A) = -\sum_{a=1}^{m} \gamma^a A \rho_{\gamma*}(X_a), \qquad A \in \mathrm{Hom}(V_\gamma, \Sigma_m)_K$$

The eigenvalue problem for the Dirac operator

$$D\Psi = \lambda\Psi$$

reduces to the eigenvalue problem in the subspaces

$$\mathrm{Hom}(V_\gamma, \Sigma_m)_K$$

Simple algebraic manipulations give the spectrum of the Dirac operator on the two-dimensional sphere to be (Cahen and Gutt, 1988)

$$\lambda_j = \pm\left(j + \frac{1}{2}\right), \qquad j = \frac{1}{2}, \frac{3}{2}, \ldots$$

3. SPINOR STRUCTURES AND DIRAC OPERATOR

In this section, we will discuss the notion of spinor structures and the connected Dirac operators on quantum spaces. We will limit ourselves to the classical spinor structures as they were defined in Đurđevich (1994) within the general framework of framed principal bundles (Đurđevich, 1999), which incorporates generalization to the case of quantum spaces of both the frame bundles and their coverings, including spin structures. In this paper, we do not go beyond the classical spin structures, so that the space of spinors on the quantum sphere is exactly the same as in the case of a classical two-dimensional sphere, i.e., $\mathbb{C}^2$. However, a more general approach is also possible, and will be studied systematically in future publications. In the more general framework (Đurđevich, 1999), the space of spinors is also treated in the "quantum" framework, and very exotic features can appear, for example, the space of spinors can become infinite dimensional, significantly complicating the whole formalism. We believe that both approaches to spinor structures and the Dirac operators on quantum spaces make sense and should be extensively studied. Lack of space keeps us from going into the full definition of the classical spin structure on a quantum space (a general one, not necessarily corresponding to a classical symmetric space only) and the appropriate Dirac operator. The complete definition can be found in Đurđevich (1994). Let us only recall that an important role in the definition is played by the notion of a frame structure, which assumes existence of a kind of horizontal vector field on the principal quantum bundle, the structure group of which happens to be a classical Spin group. The horizontal vector fields should satisfy axioms telling us that they are derivations acting in the "algebra of smooth functions" on the base noncommutative space. The latter space is just defining the noncommutative "manifold," so that it is defined in the very beginning. In particular, in the case of interest for us of the Podleś sphere, which is a quotient space for the quantum group $S_\mu U(2)$ (at least in the case of the second deformation parameter $c = 0$), just the quantum Hopf bundle with the $U(1)$ as the structure group is the classical spin structure for this noncommutative sphere (see Đurđevich, 1996, 1997, for more extensive discussion). The general form of the Dirac operator on a quantum space equipped with a frame structure (which means a set of derivatives $X_1, X_2, \dots, X_n$ satisfying a number of conditions given by Đurđevich, 1994), and with the corresponding classical Dirac matrices $\gamma_1, \gamma_2, \dots, \gamma_n$, is as follows:

$$D = X_1 \otimes \gamma_1 + X_2 \otimes \gamma_2 + \dots + X_n \otimes \gamma_n$$

and this form is used further in this paper. Since the derivation of spectra of the Dirac operators on classical symmetric (compact) spaces is reducible to algebraic calculations with irreducible (unitary) representations of appropriate Lie groups, following from the Cartan theory of such spaces, it would be interesting to find an appropriate definition of quantum symmetric spaces, which then would allow for a similar treatment of the spectral problem for the Dirac operators. This seems quite hopeless for now. Instead, one can consider just spaces corresponding to the symmetric ones, e.g., as their quantum deformations, or just quotient spaces of some quantum groups, instead of the classical groups. From Cartan's theory of symmetric spaces, it follows that symmetric spaces as a rule could be represented as quotient spaces of Lie groups, as a result of their homogeneity. Lack of Riemannian structure on quantum spaces and a lack of points, and, as a result, infinitesimal neighborhoods of the points, makes it impossible to define a notion corresponding to a homogeneous Riemannian space. There are two alternatives. We can define a homogeneous space for a quantum group as an orbit in its action on some space, or, and this is the approach we adopt in this paper (but we would like to study in the future, in particular in the context of spectral geometry, also the first option), as a quotient space of a quantum group by its subgroup, understood as such in the framework of quantum groups. This naturally introduces a quantum principal bundle structure, with the numerator quantum group of the quotient as the total space of the bundle and with the quantum group subgroup in the denominator as the structure group, as was shown in Đurđevich (1997).

4. SPECTRUM OF THE DIRAC OPERATOR ON THE PODLEŚ SPHERE

In this section, we define the Dirac operator on the Podleś sphere and find its spectrum. The Dirac operator can be written

$$D = X_+ \otimes \gamma_+ + X_- \otimes \gamma_-$$

$$\gamma_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \gamma_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the operators X_+ and X_- are part of the left-covariant differential calculus defined on the $S_\mu U(2)$ by Woronowicz (1987), which in the appropriate orthonormal bases of the $S_\mu U(2)$ unitary irreducible representation (unirrep) spaces, can be written

$$X_+|j, m\rangle = \mu^{-(m+j)}[(j + m + 1)_\mu (j - m)_\mu]^{1/2}|j, m + 1\rangle$$

$$X_-|j, m\rangle = \mu^{1-m-j}[(j + m)_\mu (j - m + 1)_\mu]^{1/2}|j, m - 1\rangle$$

$$n_\mu = \frac{1 - \mu^{2n}}{1 - \mu^2}$$

and the unirreps of $S_\mu U(2)$ are classified by $j = 0, 1/2, 1, 3/2, \ldots$, just as in the case of the classical $SU(2)$ group. The operators X_+, X_- define a frame structure on the quantum sphere. Therefore, the Dirac operator written above is aparently a self-adjoint operator and is built in the way the Dirac operators should be built on noncommutative spaces according to the general theory developed in Đurđevich (1994). The operator acts in the space of "smooth' spinor fields, which is identifiable with the space $\mathcal{B} \otimes_{U(1)} \mathbb{C}^2$, where the space $\mathbb{C}^2$ in the above tensor product is just the spinor space Σ_2, as it is in the classical sphere case, and the space $\mathcal{B}$ is the noncommutative *-algebra of "smooth functions" over the quantum group $S_\mu U(2)$, a direct generalization of the commutative *-algebra of smooth functions over the classical group $SU(2)$. The subscript $U(1)$ means we should consider equivariance classes of elements of the tensor product with respect to the natural action of the group $U(1)$. Since the space $\mathcal{B}$ is built, as in the classical case, from the functions given by irreducible representations of the $S_\mu U(2)$, the further considerations are in the proper subspaces connected with the unirreps of $S_\mu U(2)$. As in the classical case, only half-integers j contribute to the spectrum. The latter is equal to

$$\lambda_j = \pm\frac{\mu^{i+1/2} - \mu^{-(j+1/2)}}{\mu - \mu^{-1}}$$

where $\mu \neq 1, -1$. It is easy to check, on the other hand, that in the limit when μ goes to 1, we get back the previous result for the spectrum on the classical sphere. It is not difficult to check, taking into account that the degeneracy of the jth eigenvalue is $2j + 1$, in accordance with the dimensions of the jth representation spaces, that for a large number N of the eigenvalue λ_N, it changes asymptotically as $\mu^{-\sqrt{N/2}}$, which does not agree for $\mu \neq 1$ with the powerlike dependence from Connes' axiom on the asymptotic dependence on N for the eigenvalues of the Dirac operator on noncommutative spaces.

5. SUMMARY AND PROSPECTS FOR FURTHER RESEARCH

The paper is a short presentation of the spectral problem for the Dirac operator on the Podleś sphere, treated as a quotient quantum space. More extensive presentation of the mathematical subtleties of the operator (e.g., proof of its being of trace-class) and of the spectral problem will be presented elsewhere. The results can be extended in a number of ways. We would like to study in the future the case of nonclassical spinor structures (Đurđevich, 1999). Next is the extension to other quotient and also homogeneous quantum spaces. We would also like to derive physical consequences of these results, e.g., in the physics of synthetic metals (Makaruk, 1995, 1996) or in superconductivity (Owczarek, 1995). We would like to understand the reason for the disagreement of the derived spectrum with Connes' axiom, and, after studying more examples, probably suggest a revision to the axiom.

ACKNOWLEDGMENTS

The author would like to thank the organizers of the Fifth International Conference on Clifford Algebras and their Applications, in particular Prof. Zbigniew Oziewicz, for hospitality, Dr. Micho Đurđevich for extremely helpful discussions of the key points of this paper and information about his newest results, and Dr. Hanna Makaruk for numerous discussions concerning the mathematics of the Dirac operators and physical applications of spectra of the operators on symmetric spaces.

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